

# (1+1)-dimensional turbulent and chaotic systems reduced from (2+1)-dimensional Lax integrable dispersive long wave equation

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## Abstract

After extending the Clarkson-Kruskal's direct similarity reduction ansatz to a more general form, one may obtain various new types of reduction equations. Especially, some lower dimensional turbulent systems or chaotic systems may be obtained from the general type of similarity reductions of a higher dimensional Lax integrable model. Especially, the Kuramoto-Sivashinsky equation and an arbitrary three order quasi-linear equation which includes the Korteweg de-Vries Burgers equation and the general Lorenz equation as two special cases are obtained from the reductions of the (2+1)-dimensional dispersive long wave equation system. Some types of periodic and chaotic solutions of the (2+1)-dimensional dispersive long wave equation system are also discussed.

## 1 Introduction

To reduce a higher dimensional nonlinear physical model to some lower dimensional ones is one of the most important approaches in the study of nonlinear science. Usually one use the standard Lie group approach to reduce a higher dimensional partial differential equation (PDE) to lower dimensional ones[1]. Lately, the so-called nonclassical Lie group analysis is established to find lower dimensional similarity reductions[2]. To find some lower dimensional reductions by using the classical and nonclassical Lie group approaches, one has to use some tedious algebraic procedures. In the past decade, to avoid the tedious algebraic calculation in the finding of

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the similarity reductions, a simple direct powerful method is developed[3, 4]. Using the direct method, various new similarity reductions of many physical models are found though these reductions can also be obtained lately from the nonclassical Lie group approach[5, 6, 7]. In [8], the direct method is extended to find some types of conditional similarity reductions which have not yet been obtained by means of the present classical Lie group approach and nonclassical Lie group approach. In this paper, we try to extend the direct method in another direction to find lower dimensional reductions that may not be obtained by using the present classical and nonclassical Lie group approaches.

In the next section, we discuss the general aspect on the direct reduction method. In section 3, the (2+1)-dimensional dispersive long wave equation (DLWE) is used as a concrete example to realize new reduction idea and to find some new lower dimensional reductions. In section 4, we use some numerical solutions of the lower dimensional reduction models to discuss some types of exact solutions of the (2+1)-dimensional DLWE. The last section is a short summary and discussion.

## 2 General reduction ansatz of direct method

To reduce many types of (n+1)-dimensional nonlinear PDEs,

$$\Delta(x_i, u, u_{x_i}, u_{x_i x_j}, \dots, i, j = 0, 1, \dots, n) \equiv \Delta[u] = 0, \quad (x_0 \equiv t), \quad (1)$$

it is proven that the special ansatz

$$\begin{aligned} u &= \alpha(x_0, x_1, \dots, x_n) + \beta(x_0, x_1, \dots, x_n)w(\xi_0, \xi_1, \dots, \xi_{n-1}), \\ \xi_i &= \xi(x_0, x_1, \dots, x_n), \end{aligned} \quad (2)$$

is sufficient instead of

$$u = U(x_0, x_1, \dots, x_n, w(\xi_0, \xi_1, \dots, \xi_{n-1})), \quad (3)$$

where  $w(\xi_j) \equiv w(\xi_0, \xi_1, \dots, \xi_{n-1})$  satisfies an  $n$ -dimensional PDE. In (2) and (3),  $u$  and  $w$  may be some multi-component fields. However, to reduce a higher dimensional PDE to some lower dimensional ones one may use some more general ansatzes instead of (3). For instance, the ansatz (3) may be extended as

$$u = U(x_i, w_{\xi_j}, w_{\xi_{j_1} \xi_{j_2}}, \dots, w_{\xi_{j_1} \xi_{j_2} \dots \xi_{j_k}}) \equiv U[w]. \quad (4)$$

In other words, some types of derivatives of the reduction function may be included in the primary reduction ansatz. However, to find some concrete results is quite difficult by using the

general ansatz (4). By using the similar procedure of the simplification from (3) to (2) for many types of significant mathematical physics models, one may simplify (4) to

$$u = U_0[w] + U_1[w]w_{\xi_{j_1}\xi_{j_2}\dots\xi_{j_k}}, \quad (5)$$

where  $w_{\xi_{j_1}\xi_{j_2}\dots\xi_{j_k}}$  is one of the highest derivatives of  $w$  included in (4) while  $U_0[w]$  and  $U_1[w]$  are  $w_{\xi_{j_1}\xi_{j_2}\dots\xi_{j_k}}$  independent.

### 3 Special new reductions of the (2+1)-dimensional DLWE

To give out some concrete results from above general reduction ansatz, we take the (2+1)-dimensional dispersive long wave equation (2DDLWE)

$$u_{yt} + \eta_{xx} + u_x u_y + uu_{xy} = 0, \quad (6)$$

$$\eta_t + u_x + \eta u_x + u \eta_x + u_{xxy} = 0 \quad (7)$$

as a simple example. The equation system (6) and (7) is first obtained by Boiti *et al.* [9] as a compatibility condition for a ‘weak’ Lax pair. The infinite dimensional Kac-Moody-Virasoro type symmetry structure of the model is revealed by Paquin and Winternitz[10]. The more general  $W_\infty$  symmetry is given in[11]. It is proven that[12] the 2DDLWE system is fails in passing the Painlevé test both at the WTC’s (Weiss-Tabor-Carnevale) [13] meaning and at the ARS’s (Ablowitz-Ramani-Segur) meaning[14]. Using the special ansatz (2), nine types of two dimensional similarity reductions and thirteen types of ODE (ordinary differential equation) reductions has been given by one of the present authors (Lou)[15].

For simplicity further, we taking the reduction ansatz (5) in a specific form,

$$u = F_1(t, y, w)w_x + F_0(w, w_x, w_{xx}) + F_2(w, w_x)w_{xxx}, \quad (8)$$

$$v \equiv \eta + 1 = u_y = F_{1y}(t, y, w)w_x, \quad w \equiv w(x, t). \quad (9)$$

The reason why we take the ansatz (8) is that we try to find the reduction equations have the following three order autonomous PDE form

$$w_t = \alpha w_{xxx} + F_3(w, w_x, w_{xx}) \quad (10)$$

for some possible functions  $F_3$ . The ansatz (9) degenerates two equations (6) and (7) to a same one.

Substituting (8)-(10) into (6) and/or (7) yields

$$F_{1y}(t, y, w)(F_2(w, w_x)w_x + \alpha)w_{xxxx} + f(t, y, w, w_x, w_{xx}, w_{xxx}) = 0, \quad (11)$$

where  $f(t, y, w, w_x, w_{xx}, w_{xxx}) \equiv f$  is a complicated expression of the indicated variables. Because of  $f$  is  $w_{xxxx}$  independent, (11) is valid only for

$$F_2(w, w_x) = -\alpha w_x^{-1}. \quad (12)$$

Substituting (12) into (11), we have

$$F_{1y}(t, y, w)(F_{3w_{xx}}(w, w_x, w_{xx}) + 1 + w_x F_{0w_{xx}}(w, w_x, w_{xx}))w_{xxx} + f_1(t, y, w, w_x, w_{xx}) = 0, \quad (13)$$

where  $f_1(t, y, w, w_x, w_{xx}) \equiv f_1$  is independent of  $w_{xxx}$ . From eq. (13) we immediately have

$$F_3(w, w_x, w_{xx}) = -w_{xx} - w_x F_0(w, w_x, w_{xx}) + F_3(w, w_x). \quad (14)$$

By using Eq. (14), (13) is simplified further to

$$(2w_x F_{1y}(t, y, w) F_1(t, y, w) + F_{1y}(t, y, p) F_{3w_x}(w, w_x) + 2w_x F_{1yw}(t, y, p))w_{xx} + f_2(t, y, w, w_x) = 0, \quad (15)$$

where  $f_2(t, y, w, w_x) \equiv f_2$  is  $w_{xx}$  independent. Integrating (15) once with respect to  $y$ , we have

$$\left( w_x F_1(t, y, w)^2 + F_1(t, y, w) F_{3w_x}(w, w_x) + 2w_x F_{1w}(t, y, w) + f_3(t, w, w_x) \right) w_{xx} + f_2(t, y, w, w_x) = 0, \quad (16)$$

with  $f_3(t, w, w_x)$  being an integrating function. Because of the  $w_x$  independence of  $F_1(t, y, w)$ , by vanishing the first term of (16), we get

$$F_3(w, w_x) = F_{32}(w)w_x^2 + F_{30}(w), \quad f_4(t, w, w_x) = w_x F_4(t, w), \quad (17)$$

and

$$2F_{1w}(t, y, w) + F_1(t, y, w)^2 + 2F_{32}(w)F_1(t, y, w) + F_4(t, w) = 0. \quad (18)$$

Because of (17) and (18), Eq. (16) is simplified finally to

$$F_{1t}(t, y, w) + F_1(t, y, w)F_{30w}(w) - \frac{1}{2}F_{30}(w)(F_1(t, y, w) + 2F_{32}(w))F_1(t, y, w) + F_5(t, w) = 0. \quad (19)$$

The compatibility condition of (18) and (19) requires that

$$F_5(t, w) = -F_{30ww}(w) + F_{32}(w)F_{30w}(w) - \frac{1}{2}F_4(t, w)F_{30}(w) + F_{32w}(w)F_{30}(w) \quad (20)$$

and

$$\begin{aligned} & F_4(t, w)F_{30w}(w) - F_{32}(w)F_{30}(w)F_{32w}(w) - 2F_{30w}(w)F_{32w}(w) + \frac{1}{2}F_{30}(w)F_{4w}(t, w) \\ & - F_{32ww}(w)F_{30}(w) - F_{32}(w)^2F_{30w}(w) + F_{30www}(w) + \frac{1}{2}F_{4t}(t, w) = 0. \end{aligned} \quad (21)$$

Now the final results show us that the 2DDLWE (6) and (7) possesses the following reduction

$$w_t = \alpha w_{xxx} - w_{xx} - w_x F_0(w, w_x, w_{xx}) + F_{32}(w)w_x^2 + F_{30}(w) \quad (22)$$

with arbitrary functions  $F_0(w, w_x, w_{xx})$ ,  $F_{32}(w)$  and  $F_{30}(w)$  and

$$u = -\alpha w_x^{-1} w_{xxx} + F_0(w, w_x, w_{xx}) + F_1(t, y, w)w_x, \quad (23)$$

$$v = \eta + 1 = F_{1y}(t, y, w)w_x, \quad (24)$$

where  $F_1(t, y, w)$  is determined by two compatible Riccati equations (18) and (19) while  $F_4(t, w)$  and  $F_5(t, w)$  are determined by (20) and (21). The simplest solution of (18)–(21) reads

$$F_4(t, w) = F_{32}(w) = F_5(t, w) + 2A_2 = 0, \quad (25)$$

$$F_{30}(w) = A_2 w^2 + A_1 w + A_0, \quad (26)$$

$$F_1(t, y, w) = \frac{2}{w + q(y, t)}, \quad (27)$$

$$q_t(y, t) = A_1 q(y, t) - A_0 - A_2 q(y, t)^2, \quad (28)$$

where  $A_0$ ,  $A_1$ ,  $A_2$  are arbitrary constants.

From the reduction equation (24), we can see that though the original 2DDLWE system is Lax integrable, possesses infinitely many symmetries and abundant multi-soliton structures, there still exist various nonintegrable lower dimensional reductions because of the entrance of three arbitrary functions  $F_0(w, w_x, w_{xx})$ ,  $F_{32}(w)$  and  $F_{30}$ . For instance, if we select  $F_0(w, w_x, w_{xx})$ ,  $F_{32}(w)$  and  $F_{30}(w)$  simply as

$$-w_x F_0(w, w_x, w_{xx}) + F_{32}(w)w_x^2 + F_{30}(w) = ww_x, \quad (29)$$

then (22) becomes the well known KdV-Burgers equation

$$w_t = \alpha w_{xxx} - w_{xx} + ww_x \quad (30)$$

which is one of the possible candidate to describe the turbulence phenomena in fluid physics and plasma physics[16, 17]. If the functions  $F_0(w, w_x, w_{xx})$ ,  $F_{32}(w)$  and  $F_{30}$  are fixed to satisfy

$$\begin{aligned} & -w_x F_{0w_{xx}}(w, w_x, w_{xx}) + F_{32}(w)w_x^2 + F_{30}(w) \\ &= \frac{1}{w}[w_{xx}w_x + (c+1)w_x^2] - w^2w_x - (b+c)w_{xx} - wc(b-ba+w^2) \end{aligned} \quad (31)$$

with  $\alpha = -1$  and  $a$ ,  $b$ ,  $c$  are arbitrary constants, then (24) becomes a (1+1)-dimensional extension

$$w_t = -w_{xxx} - (b+c+1)w_{xx} + \frac{1}{w}[w_{xx}w_x + (c+1)w_x^2] - w^2w_x - wc(b-ba+w^2) \quad (32)$$

of the famous chaotic system, the Lorenz system[18]

$$w_s = -c(w - g), \quad g_s = (a - h)w - g, \quad h_s = wg - bh. \quad (33)$$

Actually, the travelling wave reduction of (32),  $w = w(x + b(c+1)t) \equiv w(s)$ , is totally equivalent to the Lorenz system (33).

In principle, any order of derivatives of  $w$  may be included in the ansatz (5). And some types of more complicated reduction equations can be obtained. For instance, if we insert a fourth order derivative  $w_{xxxx}$  term into the reduction ansatz, we may obtain many fourth order (1+1)-dimensional PDE reductions. Here we list only a special example for the reduction equation has a famous Kuramoto-Sivashinsky (KS) equation form[19]

$$\begin{aligned} u &= \pm \frac{2w_x(a_1 + a_3Q)}{a_0 + a_1w + a_2Q + a_3wQ} - \frac{(\alpha_3 \mp 1)w_{xx}}{w_x} + \alpha_1w \\ &\quad + \frac{1}{w_x}(\alpha_5w_{xxxx} + (a_1c_2 + a_3c_0)w^2 + (c_2a_0 + c_1a_1 + a_2c_0 + \alpha_2)w + c_1a_0), \end{aligned} \quad (34)$$

$$\eta = \frac{Q_y p_x(a_3a_0 - a_2a_1)}{a_0 + a_1p + a_2Q + a_3pQ)^2}, \quad (35)$$

$$Q_t = a_0c_0 + (c_1a_1 + a_2c_0 - c_2a_0)Q + (c_1a_3 - c_2a_2)Q^2, \quad (36)$$

$$w_t + \alpha_1ww_x + \alpha_2w + \alpha_3w_{xx} + \alpha_4w_{xxxx} = 0, \quad (37)$$

where  $a_i$ ,  $\alpha_i$ ,  $i = 0, 1, \dots, 5$  are arbitrary constants and  $c_0$ ,  $c_1$  and  $c_3$  are arbitrary functions of  $t$ . Various interesting properties of the chaotic KS equation (37) have been studied by many authors, say, [19] and the references therein.

In the reduction results (22) and (37), the independent variables are simply taken as  $x$  and  $t$ . Actually, extending these independent variables to some more general forms is possible because the model possesses infinitely many symmetries with some arbitrary functions[10, 11]. For instance, using the finite transformations given in [10], all the independent variables of the systems (22) and (37) are changed to some general forms naturally.

## 4 Special solutions

Now an interesting question is which kinds of exact solutions can be obtained from our new reduction equations? In this section, we write and plot down some interesting exact solutions.

### 4.1 Multi-dromion solutions

If we take  $F_0(w, w_x, w_{xx})$  as

$$F_0(w, w_x, w_{xx}) = 6w - w_x^{-1}(w_{xx} - w_x^2 F_{32}(w) - F_{30}(w)), \quad (38)$$

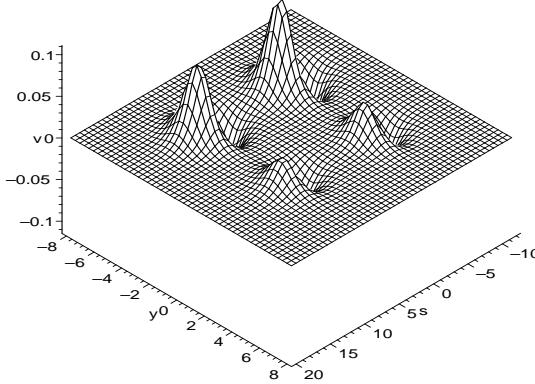


Figure 1: Four dromion solution of the 2DDLWE for the field  $v = \eta + 1$  given by (24) with (27) and (41)–(44).

then we know that the  $w$  equation (22) is just the well known KdV equation

$$w_t = \alpha w_{xxx} - 6ww_x. \quad (39)$$

Then we can use the  $N$  soliton solutions of the (1+1)-dimensional KdV equation to construct the multi-dromion solutions by taking

$$q(y, t) = a_0 + \sum_{n=1}^N a_n \tanh(l_n y - y_n) \quad (40)$$

with  $a_i$ ,  $i = 0, 1, \dots, N$  being arbitrary constants and  $A_0 = A_1 = A_2 = 0$ . Fig. 1 is a plot of the four dromion solution with  $w$  being two soliton solution of the KdV equation

$$w = -2\alpha(\ln \phi)_{xx}, \quad (41)$$

$$\begin{aligned} \phi &= 1 + \exp(k_1 x + \alpha k_1^3 t + x_1) + \exp(k_2 x + \alpha k_2^3 t + x_2) \\ &\quad + \frac{(k_1 - k_2)^2}{(k_1 + k_2)^2} \exp((k_1 + k_2)x + \alpha(k_1^3 + k_2^3)t + x_1 + x_2) \end{aligned} \quad (42)$$

and

$$q = 4 + \tanh(l_1 y - y_1) + 2 \tanh(l_2 y - y_2) \quad (43)$$

while the other constants are fixed as

$$k_1 = 1, k_2 = 1.1, l_1 = l_2 = 1, \alpha = -1, x_1 = -3, x_2 = 3, y_1 = -3, y_2 = 3. \quad (44)$$

The figure (1) and all other figures of this paper are plotted at time  $t = 0$ .

## 4.2 Periodic and chaotic line soliton solutions

If  $q$  is still given by (43) while  $w$  is given by (33), then we may obtain some kinds of periodic or chaotic line soliton solutions. Because no one has given ever out any exact explicit solutions of

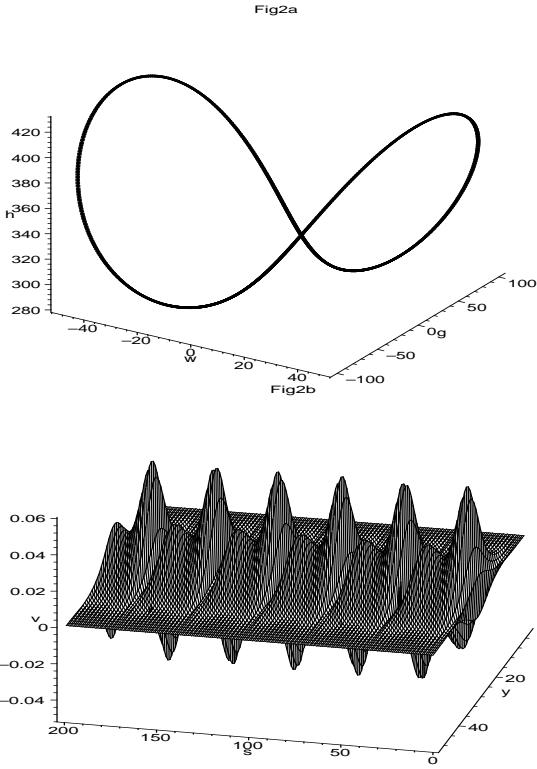


Figure 2: (a) Periodic two solution of the Lorenz system. (b) The related periodic two line soliton solution of the 2DDLWE for the field  $v$ .

(33), we can only use the numerical solutions of the generalized Lorenz system to construct exact solutions of the 2DDLWE. For several types of parameter ranges, the solutions of the Lorenz system are periodic while for other types of parameter ranges, the solutions of the Lorenz system are chaotic ones. Fig. 2 is a plot of the periodic two line soliton solution of the 2DDLWE with the parameters of the Lorenz system (33) is fixed as

$$a = 350, \ b = \frac{8}{3}, \ c = 10 \quad (45)$$

and

$$q = 200 + \tanh y \quad (46)$$

From Fig. 2b we can see that the line solution is localized in  $y$  direction and periodic in  $s (= x + b(c+1)t)$  direction when the parameters are selected appropriately as (45).

Fig. 3 plots the chaotic line soliton solution of the 2DDLWE with the parameters of the Lorenz system (33) given by

$$a = 60, \ b = \frac{8}{3}, \ c = 10 \quad (47)$$

while the  $q$  function still given by (46). Obviously, Fig. 3 shows us that when the parameters

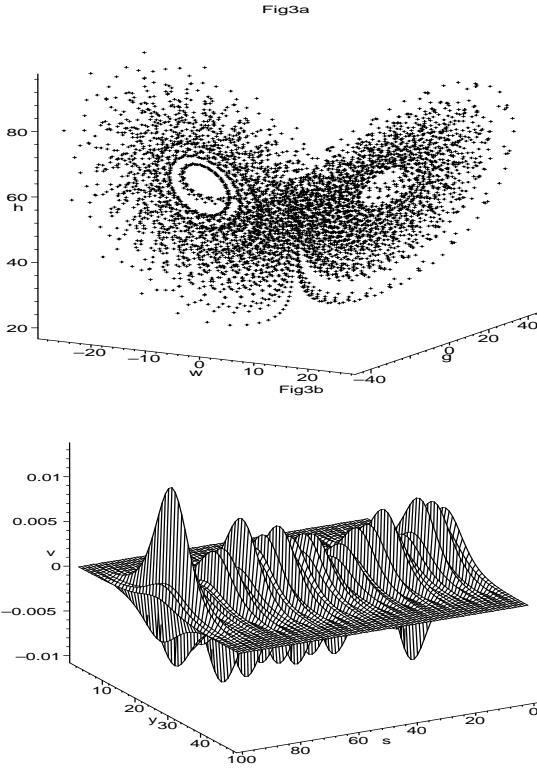


Figure 3: (a) chaotic solution of the Lorenz system. (b) The related chaotic line soliton solution of the 2DDLWE for the field  $v$ .

of (33) are located at the chaotic regions, the corresponding solution becomes a chaotic straight line soliton solution which is localized in  $y$  direction and chaotic in  $s$  direction.

### 4.3 Space periodic and chaotic solutions

From (28) we know that in some cases, the function  $q$  may be an arbitrary function of  $y$ , so we may also select it as a solutions of the Lorenz system (33) with the replacement of the independent variable  $s \rightarrow y$ . When the function  $q = q(y)$  and  $w = w(s) = w(x + b(c+1)t)$  are all the solutions of the Lorenz system, we can obtain many types of solutions which are periodic or chaotic in both directions.

Fig.4 is a plot of a periodic solution of the 2DDLWE which has periodic property in both directions while  $q(y)$  and  $w(s)$  are chosen as both the solutions of the Lorenz system (33) with (45).

Fig.5 is a plot of an exact solution of the 2DDLWE which is periodic in  $y$  direction and chaotic in  $x$  direction. The corresponding solutions for  $q$  and  $w$  are all determined by (33) but with different parameters (45) and (47) respectively.

Fig.6 shows a chaotic solution of the 2DDLWE in both directions. The related solutions for

Fig4

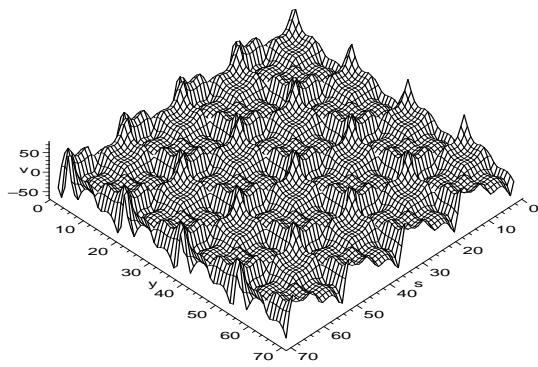


Figure 4: Periodic two solution of the 2DDLWE in both direction for the field  $v$ .

Fig5

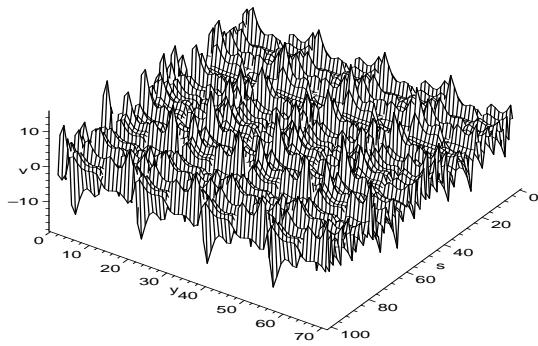


Figure 5: Plot of the exact solution of the 2DDLWE for the field which is periodic in  $y$  direction and chaotic in  $x$  direction.

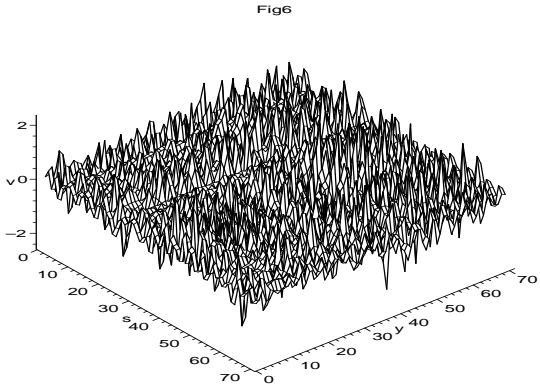


Figure 6: A typical solution of the 2DDLWE which is chaotic in both directions.

$q$  and  $w$  are all determined by (33) with the same parameters (47).

## 5 Summary and discussions

In summary, the CK's direct similarity reduction ansatz is extended to a much more general form. Using the general reduction ansatz, one may obtain various new lower dimensional reduction equations including many turbulence and chaotic systems. Taking the 2DDLWE as a concrete example, and a slightly special reduction ansatz with three order derivatives of the reduction field, we obtain a general three order quasi-linear equation, which includes the KdV, MKdV, KdV-Burgers and the generalized Lorenz system as special examples, as a special reduction of the 2DDLWE. The known KS system and other types of higher order models may also be obtained from the reductions of the 2DDLWE.

The reductions (30), (32) and (37) are known as some typical turbulence and chaotic systems while the 2DDLWE is known as an IST integrable model. The reason why some lower dimensional turbulence and chaotic systems can be reduced from a higher dimensional integrable (under some particular meanings) model is that for a higher dimensional integrable model, some types of lower dimensional arbitrary functions do enter into its general solution.

Using the solutions of the lower dimensional models we may obtain many kinds of new solutions for the 2DDLWE. Especially, using the numerical solutions of the Lorenz systems, some types of periodic line solitons, chaotic line solitons, periodic-periodic solution, periodic-chaotic and chaotic-chaotic solutions of the 2DDLWE can be obtained.

In Ref. [20], using the variable separation approach, we have also pointed out that the turbulence and chaotic systems can be obtained from other “integrable” models like the Davey-Stewartson equations and the asymmetric Nizhnik-Novikov-Veselov equation because of the entrance of the lower dimensional arbitrary functions in the general solutions. The more about

the method and the effects of the reduced turbulence system on the original model(s) are worthy of studying further.

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